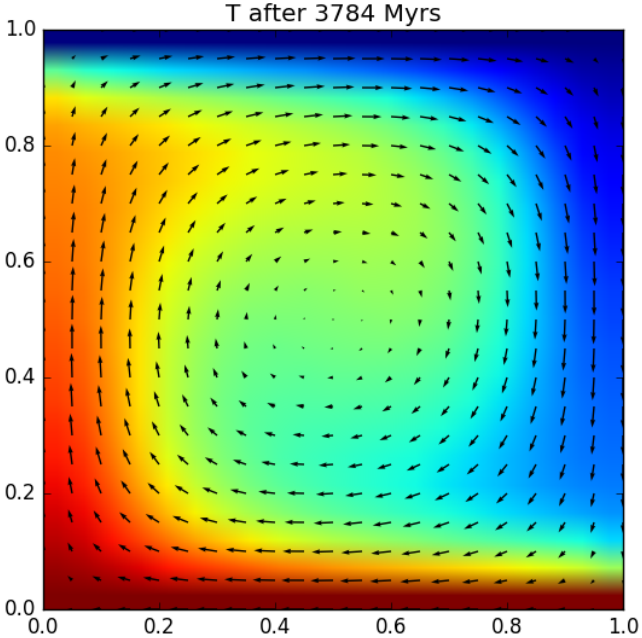
SUBITOP Understanding subduction zone topography through modelling of coupled shallow and deep processes

Solving the flow field

- Conservation of momentum & mass
- Numerical approach
- Staggered grids
- Solving matrix-vector systems
- Rayleigh-Bénard convection
- Calculating topography

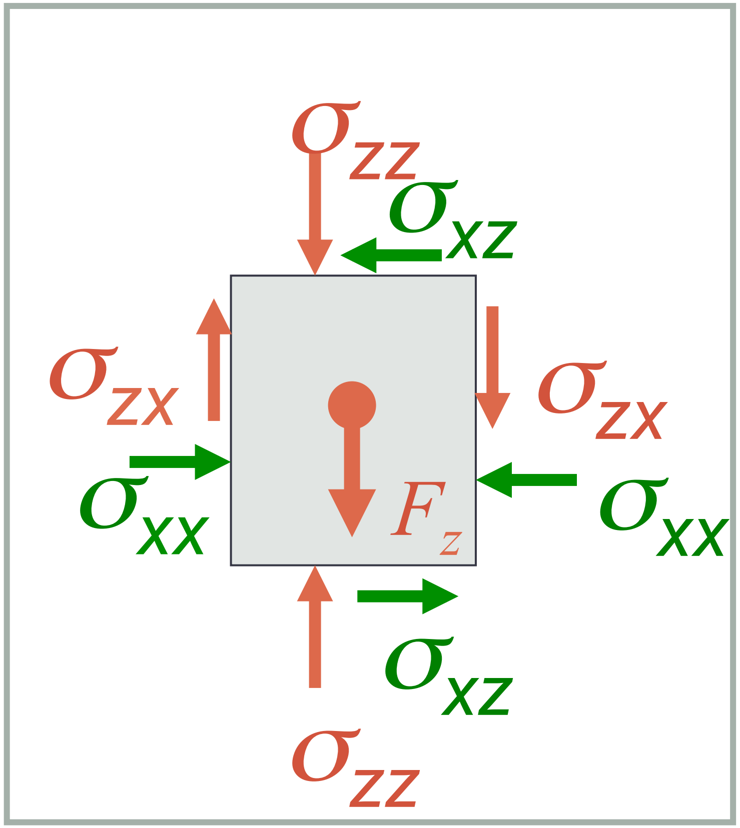


Force balance: the Stokes equation (1)

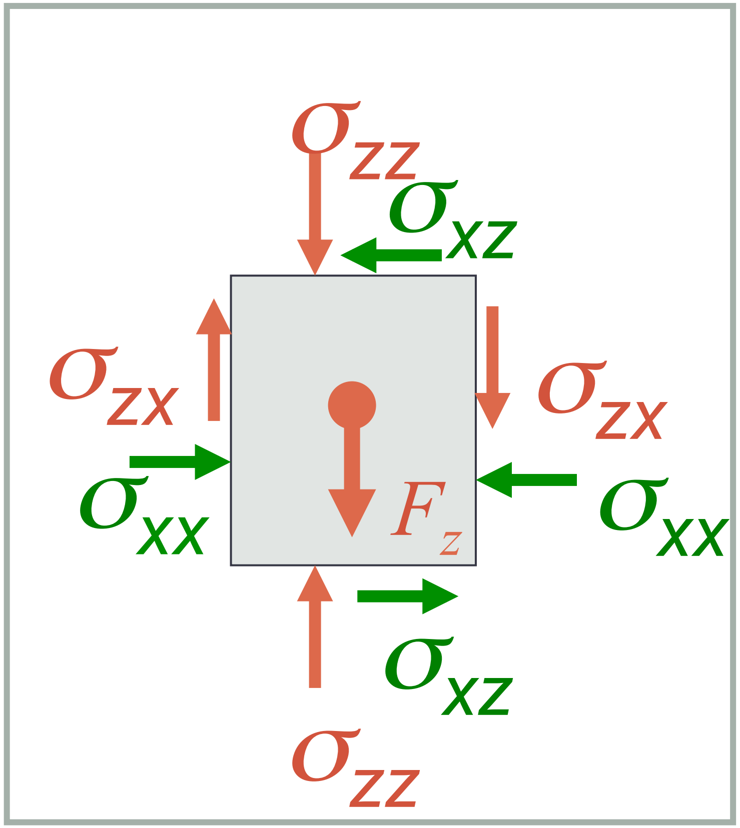
- Sum of forces equals 0:
  - difference in σ_{ij} over block
  - body force F_i

  ---------------------------------- +

  - total force = 0



Force balance: the Stokes equation (1)



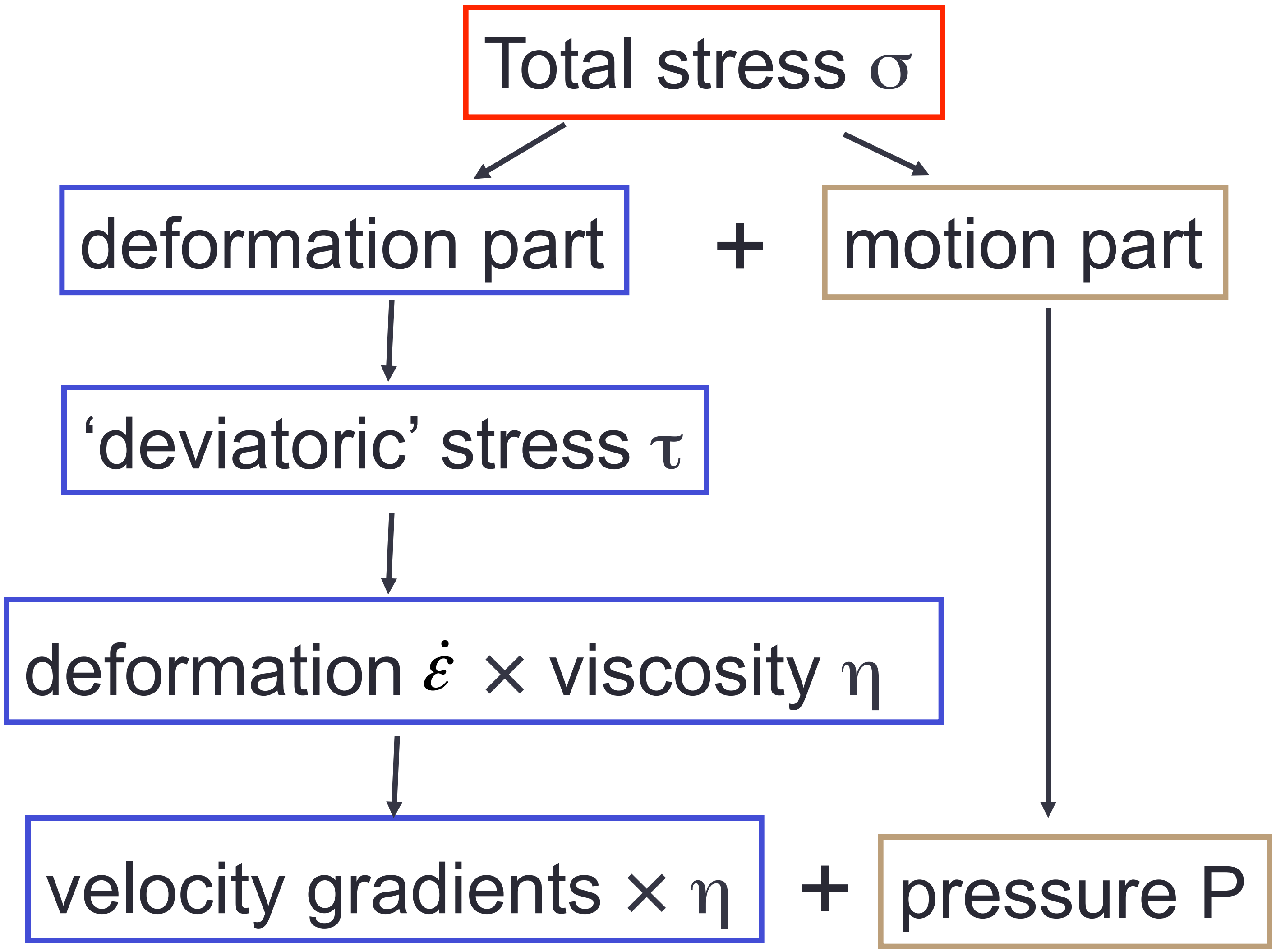
- in x-dir.: $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$
- in z-dir.: $$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + F_z = 0$$

Force balance: the Stokes equation (2)

$$\sigma = \tau - P$$

$$\tau = 2\eta\dot{\varepsilon}$$

$$\dot{\varepsilon} = \tfrac{1}{2}\nabla u$$



Force balance: the Stokes equation (3)

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \\ \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + F_z = 0 \end{aligned}\right\} \nabla \cdot \sigma + \vec{F}_z = 0$$

Using:

$$\sigma = \tau - P$$

$$\tau = 2\eta\dot{\varepsilon}$$

$$\left.\begin{aligned} \dot{\varepsilon} = \tfrac{1}{2}\nabla u \\ \eta \text{ constant} \end{aligned}\right\}$$

$$\nabla \cdot \tau - \nabla P + \vec{F}_z = 0$$

$$2\nabla \cdot (\eta\dot{\varepsilon}) - \nabla P + \vec{F}_z = 0$$

$$\eta\nabla^2 v - \nabla P + \vec{F}_z = 0 \quad \left\{\begin{aligned} \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial P}{\partial x} = 0 \\ \eta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - \frac{\partial P}{\partial z} + F_z = 0 \end{aligned}\right.$$

Force balance: the Stokes equation (4)

- in x- and z-direction:

$$\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial P}{\partial x}=0$$

$$\eta\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{\partial P}{\partial z}+F_z=0$$

So we have:

- 2 equations
- 3 unknowns:
  - pressure P
  - velocities v_x & v_z

} additional constraint required

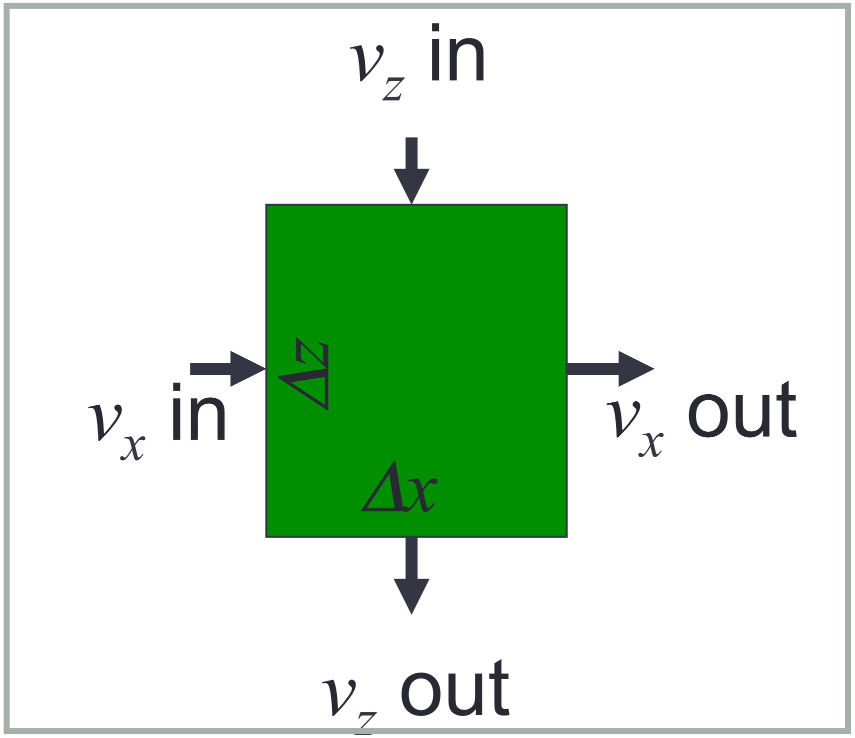
Mass conservation (continuity equation)

Net in/outflow = 0

$$\Delta z(v_{x,in} - v_{x,out}) + \Delta x(v_{z,in} - v_{z,out}) = 0$$

$$\frac{v_{x,in} - v_{x,out}}{\Delta x} + \frac{v_{z,in} - v_{z,out}}{\Delta z} = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0$$



Set of governing equations for mantle convection

Conservation of:

- momentum:

$$\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial P}{\partial x}=0$$

$$\eta\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{\partial P}{\partial z}=\alpha\rho\delta Tg$$

- mass:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0$$

- heat:

$$\frac{\partial T}{\partial t}=\kappa\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right]-v_x\frac{\partial T}{\partial x}-v_z\frac{\partial T}{\partial z}$$

Scaling and the Rayleigh number *Ra*

Scaling parameters with: $x = x'h, \; t = t'\frac{h^2}{\kappa}, \;\; T = T'\Delta T$

from which can be derived: $v = v'\frac{\kappa}{h}, \;\; P = P'\frac{\eta\kappa}{h^2}$

- momentum:

- mass:

- heat:

Scaling and the Rayleigh number *Ra*

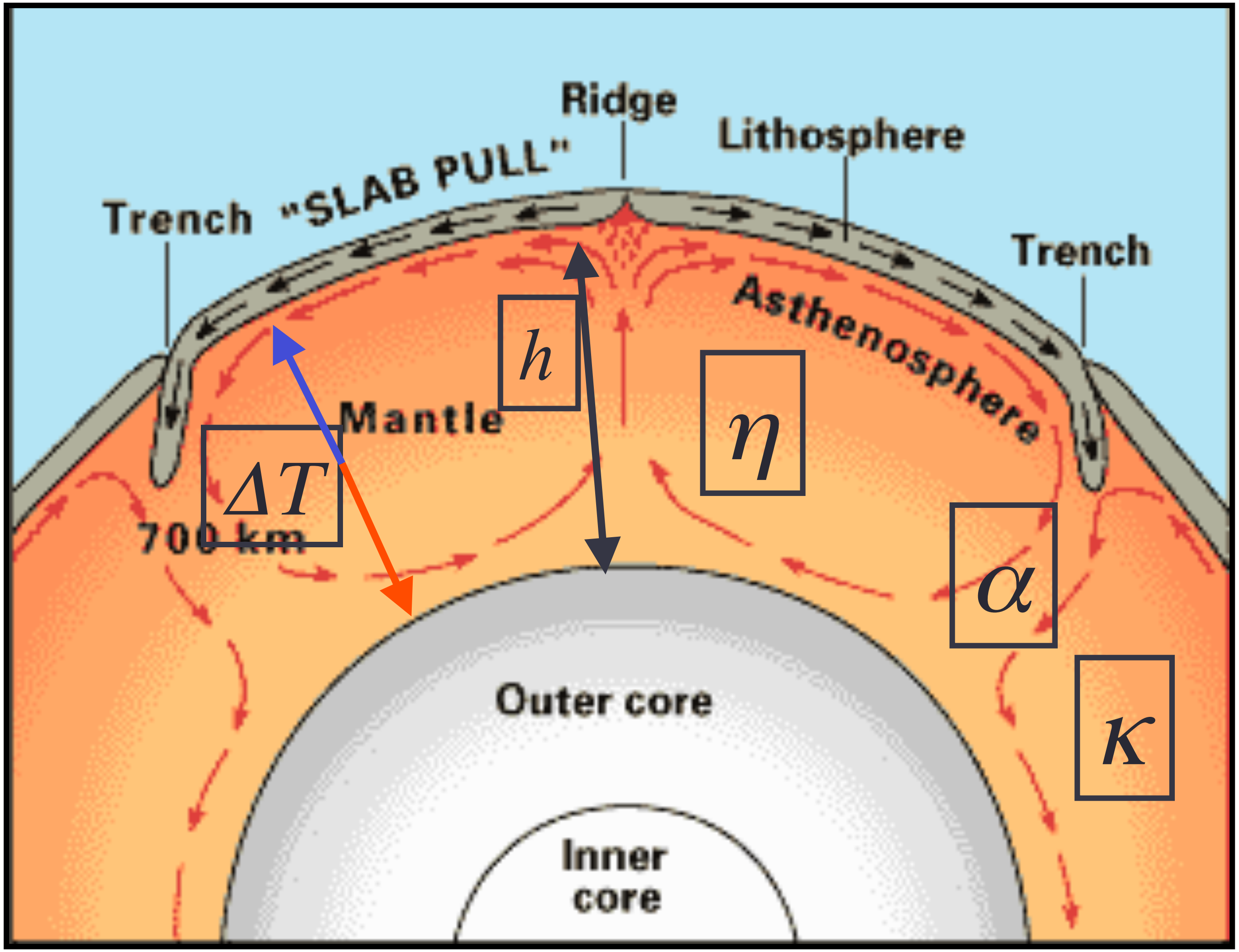
Scaling parameters with: $x = x'h,\ t = t'\frac{h^2}{\kappa},\quad T = T'\Delta T$

from which can be derived: $v = v'\frac{\kappa}{h},\quad P = P'\frac{\eta\kappa}{h^2}$

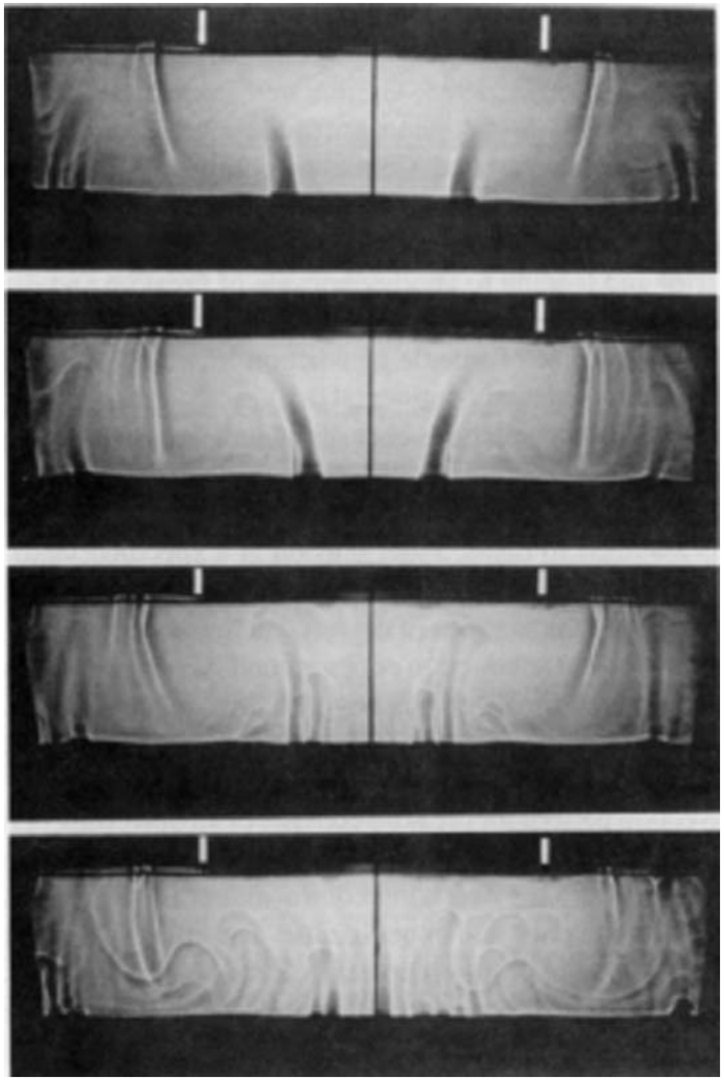
- momentum: $$\left(\frac{\partial^2 v'_x}{\partial x'^2} + \frac{\partial^2 v'_x}{\partial z'^2}\right) - \frac{\partial P'}{\partial x'} = 0$$
$$\left(\frac{\partial^2 v'_z}{\partial x'^2} + \frac{\partial^2 v'_z}{\partial z'^2}\right) - \frac{\partial P'}{\partial z'} = RaT'$$
- mass: $$\frac{\partial v'_x}{\partial x'} + \frac{\partial v'_z}{\partial z'} = 0$$
with: $$Ra = \frac{\alpha\rho g\Delta T h^3}{\eta_0\kappa}$$
- heat: $$\frac{\partial T'}{\partial t'} = \frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial z^2} - v_x\frac{\partial T'}{\partial x'} - v_z\frac{\partial T'}{\partial z'}$$

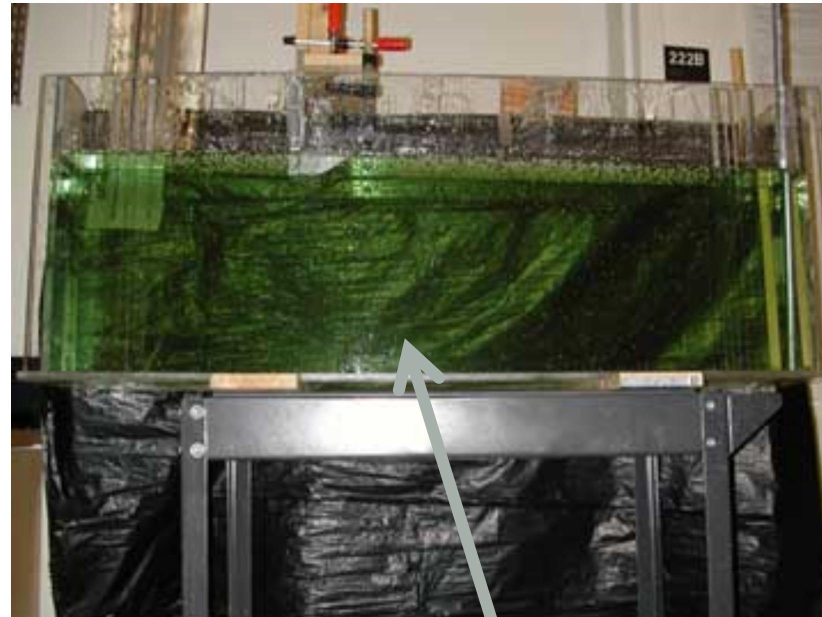
Scaling and the Rayleigh number *Ra*

$$Ra = \frac{\alpha \rho g \Delta T h^3}{\eta \kappa}$$



Analogue convection modelling





(Guillou and Jaupart,1995; geology.um.maine.edu)

Solving Stokes + continuity equations

1. $$\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial P}{\partial x} = 0$$

2. $$\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - \frac{\partial P}{\partial z} = RaT$$

3. $$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0$$

Solving the Stokes equations

Just like solving 2 equations with 2 unknowns:

$$\begin{aligned} 3x + 4y &= 19 \\ 2x + 2y &= 10 \end{aligned} \quad \rightarrow \quad \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19 \\ 10 \end{pmatrix} \quad \rightarrow \quad A\vec{x} = \vec{b}$$

We can solve the 3 * nx * nz flow equations as follows:

$$\begin{pmatrix} \text{Large matrix} \end{pmatrix} \begin{pmatrix} P^1 \\ v_x^1 \\ v_z^1 \\ P^2 \\ v_x^2 \\ v_z^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ RaT \\ 0 \\ \vdots \end{pmatrix} \quad \rightarrow \quad A\vec{x} = \vec{b}$$

Calculating observables

- Surface heat flow:

$$q_{surf} = -k \left.\frac{dT}{dz}\right|_{z=0} \approx -k \frac{T_{i,1} - T_{i,0}}{\Delta z}$$

- Free-slip surface 'topography':

$$h = -\left.\frac{\sigma_{zz}}{\Delta\rho g}\right|_{z=0} = -\left.\frac{\tau_{zz} - P}{\Delta\rho g}\right|_{z=0} = -\left.\frac{2\eta\dot{\varepsilon}_{zz} - P}{\Delta\rho g}\right|_{z=0} = -\left.\frac{2\eta\frac{dv}{dz} - P}{\Delta\rho g}\right|_{z=0}$$

$$= \frac{2\eta\frac{(v_{z,i,1} - v_{z,i,0})}{\Delta z} - P_{i,0}}{(\rho_0 - \rho_{surf})g}$$

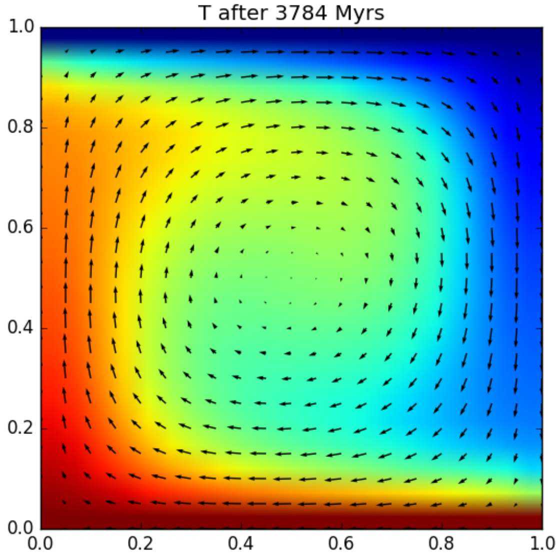
Surface topography

- Free-slip surface 'topography': $h = -\left.\frac{\sigma_{zz}}{\Delta\rho g}\right|_{z=0}$
  - Simple to calculate
  - Instantaneous topography
  - Only works well over smooth surfaces, relatively long wavelengths
- Real surface topography:
  - Allow surface nodes to move vertically and measure topography
  - Needs stabilisation of interface*
- 'Sticky-air' topography:
  - Create a non-diffusive interface with weak, light material on top of solid Earth
  - weak layer viscosity ~ $10^{18} – 10^{19}$ Pa s
  - Needs tracking and stabilisation of interface*

*(Kaus et al., 2010; Andrés-Martínez et al., 2015)

Practical 4:

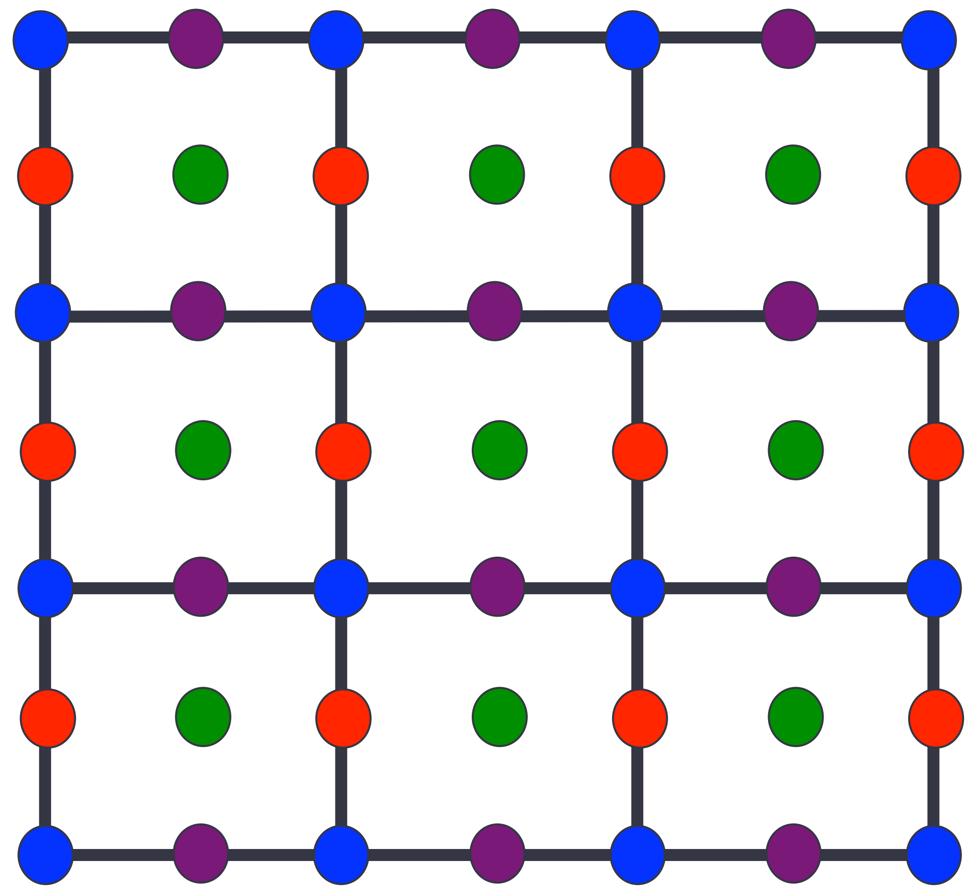
- ❑ Build a Releigh-Benard convection solver:
- ❑ Convert a provided Stokes solver into a subfuction
- ❑ Modify your advection-diffusion solver to incorporate this Stokes solver
- ❑ Calculate derived quantities like topography.



https://community.dur.ac.uk/jeroen.van-hunen/Subitop/session4.html

Extra: Staggered grids

T v_z v_x P

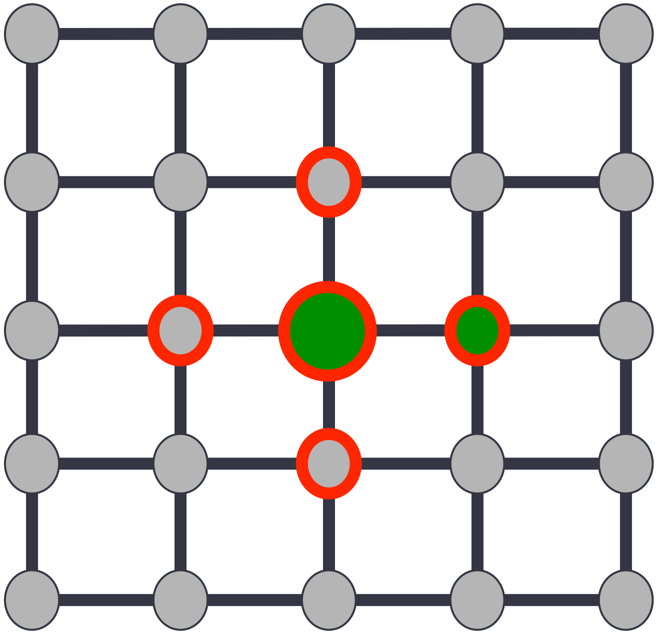


Staggered grids: Stokes-x equation

1. $$\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial P}{\partial x} = 0$$

?

$$\frac{v_{x,right} - 2v_{x,central} + v_{x,left}}{\Delta x^2} + \frac{v_{x,top} - 2v_{x,central} + v_{x,bottom}}{\Delta z^2} - \frac{P_{right} - P_{left}}{\Delta x} = 0$$



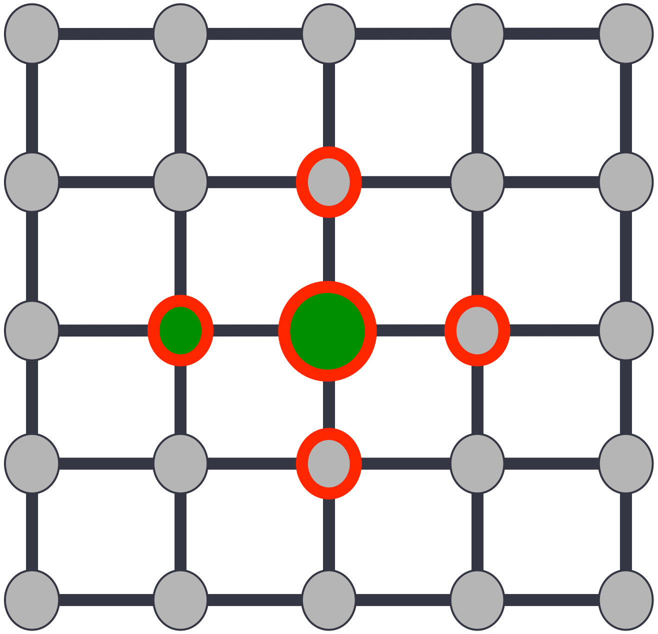
Pressure gradient discretisation not centred around central v_x point

Staggered grids: Stokes-x equation

1.
$$\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial P}{\partial x}=0$$

?

$$\frac{v_{x,right}-2v_{x,central}+v_{x,left}}{\Delta x^2}+\frac{v_{x,top}-2v_{x,central}+v_{x,bottom}}{\Delta z^2}-\frac{P_{right}-P_{left}}{\Delta x}=0$$



Pressure gradient discretisation not centred around central v_x point

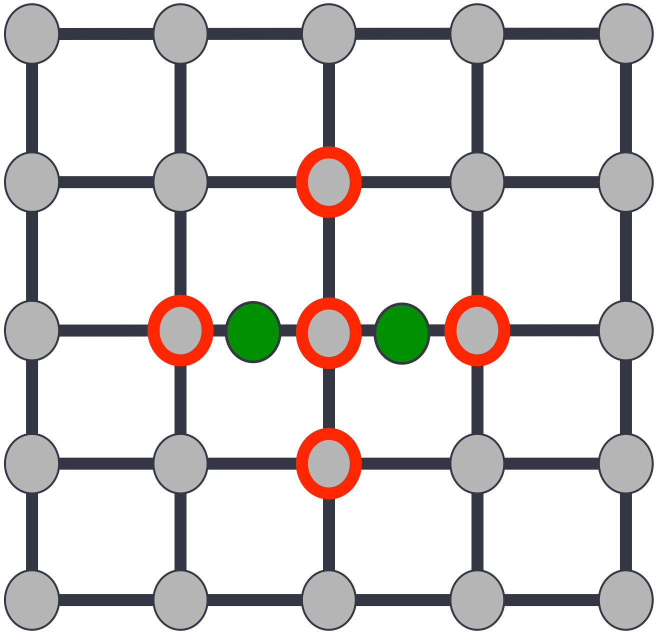
Staggered grids: Stokes-x equation

1. $$\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial P}{\partial x}=0$$

?

$$\frac{v_{x,right}-2v_{x,central}+v_{x,left}}{\Delta x^2}+\frac{v_{x,top}-2v_{x,central}+v_{x,bottom}}{\Delta z^2}-\frac{P_{right}-P_{left}}{\Delta x}=0$$

Pressure gradient discretisation is centred around central v_x point, but is not a grid point



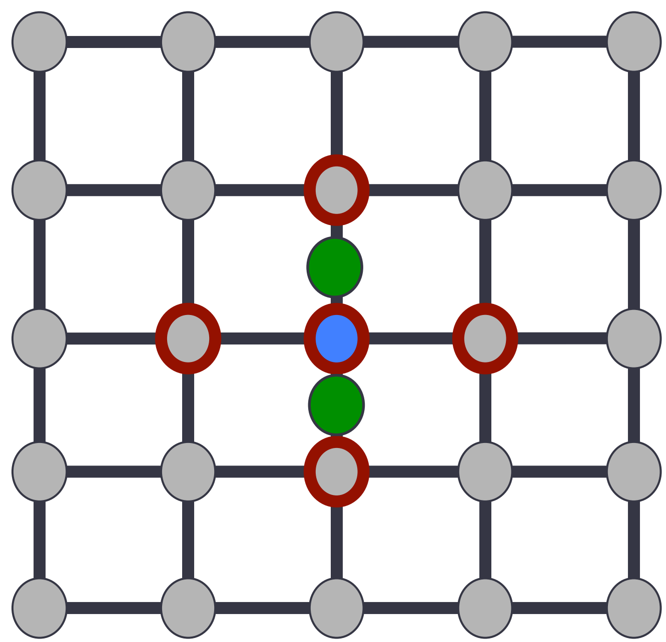
Staggered grids: Stokes-z equation

2.

$$\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{\partial P}{\partial z}=RaT$$

$$\frac{v_{z,right}-2v_{z,central}+v_{z,right}}{\Delta x^2}+\frac{v_{z,bottom}-2v_{z,central}+v_{z,top}}{\Delta z^2}-\frac{P_{bottom}-P_{top}}{\Delta z}=RaT_{central}$$

Pressure gradient discretisation is centred around central v_z point, but is not a grid point



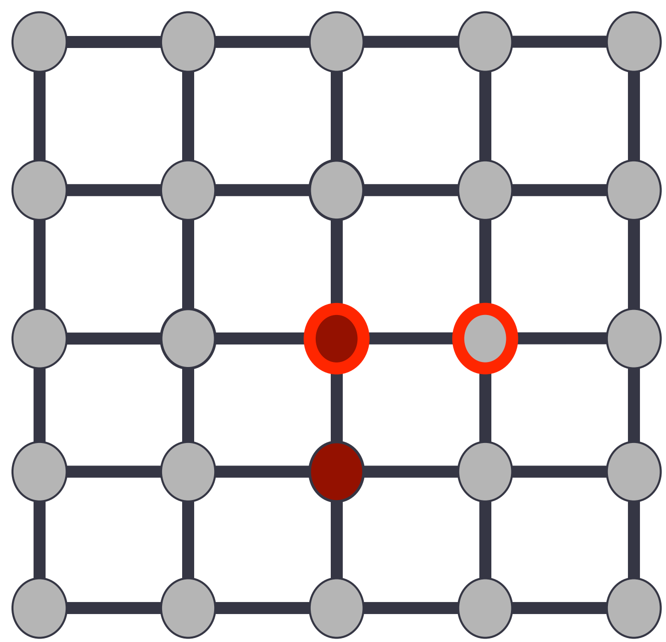
Staggered grids: continuity equation

3.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{v_{x,right} - v_{x,left}}{\Delta x} + \frac{v_{z,bottom} - v_{z,top}}{\Delta z} = 0$$

? ?



Both velocity gradients not centred around a common point

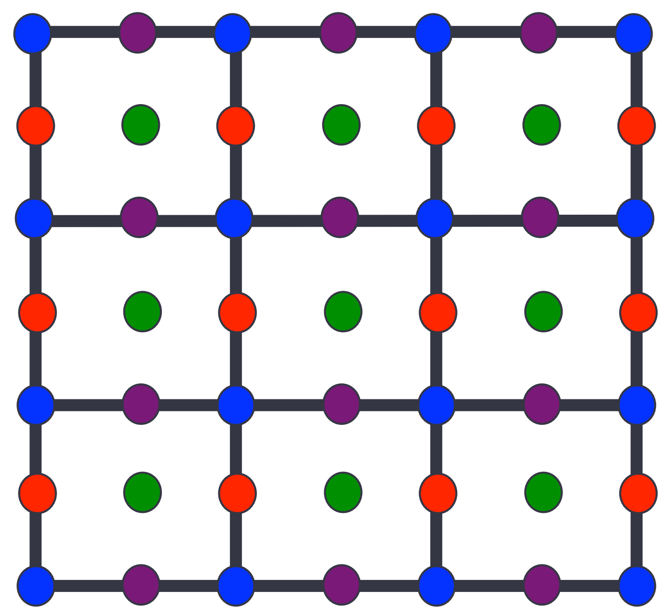
Staggered grids

$$\frac{v_{x,right} - 2v_{x,central} + v_{x,left}}{\Delta x^2} + \frac{v_{x,top} - 2v_{x,central} + v_{x,bottom}}{\Delta z^2} - \frac{P_{right} - P_{left}}{\Delta x} = 0$$

$$\frac{v_{z,right} - 2v_{z,central} + v_{z,right}}{\Delta x^2} + \frac{v_{z,bottom} - 2v_{z,central} + v_{z,top}}{\Delta z^2} - \frac{P_{bottom} - P_{top}}{\Delta z} = Ra\frac{T_{right} + T_{left}}{2}$$

$$\frac{v_{x,right} - v_{x,left}}{\Delta x} + \frac{v_{z,bottom} - v_{z,top}}{\Delta z} = 0$$

On this staggered grid, these problems disappear



T v_z v_x P

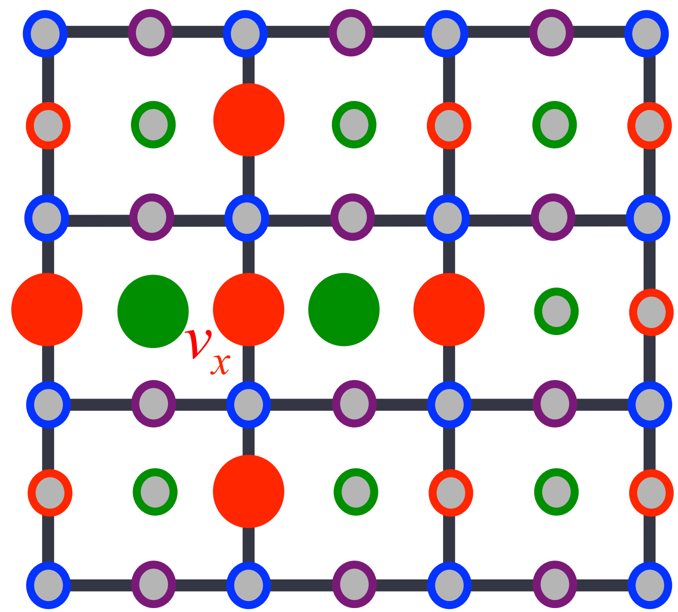
Staggered grids: Stokes-x equation

$$\frac{v_{x,right} - 2v_{x,central} + v_{x,left}}{\Delta x^2} + \frac{v_{x,top} - 2v_{x,central} + v_{x,bottom}}{\Delta z^2} - \frac{P_{right} - P_{left}}{\Delta x} = 0$$

$$\frac{v_{z,right} - 2v_{z,central} + v_{z,right}}{\Delta x^2} + \frac{v_{z,bottom} - 2v_{z,central} + v_{z,top}}{\Delta z^2} - \frac{P_{bottom} - P_{top}}{\Delta z} = Ra\frac{T_{right} + T_{left}}{2}$$

$$\frac{v_{x,right} - v_{x,left}}{\Delta x} + \frac{v_{z,bottom} - v_{z,top}}{\Delta z} = 0$$

Pressure gradient discretisation is centred around central v_x point.

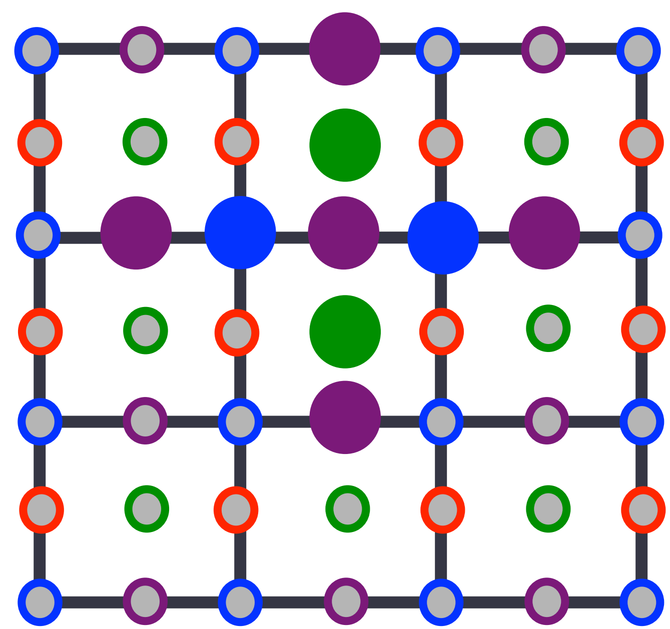


Staggered grids: Stokes-z equation

$$\frac{v_{x,right} - 2v_{x,central} + v_{x,left}}{\Delta x^2} + \frac{v_{x,top} - 2v_{x,central} + v_{x,bottom}}{\Delta z^2} - \frac{P_{right} - P_{left}}{\Delta x} = 0$$

$$\frac{v_{z,right} - 2v_{z,central} + v_{z,right}}{\Delta x^2} + \frac{v_{z,bottom} - 2v_{z,central} + v_{z,top}}{\Delta z^2} - \frac{P_{bottom} - P_{top}}{\Delta z} = Ra\frac{T_{right} + T_{left}}{2}$$

$$\frac{v_{x,right} - v_{x,left}}{\Delta x} + \frac{v_{z,bottom} - v_{z,top}}{\Delta z} = 0$$



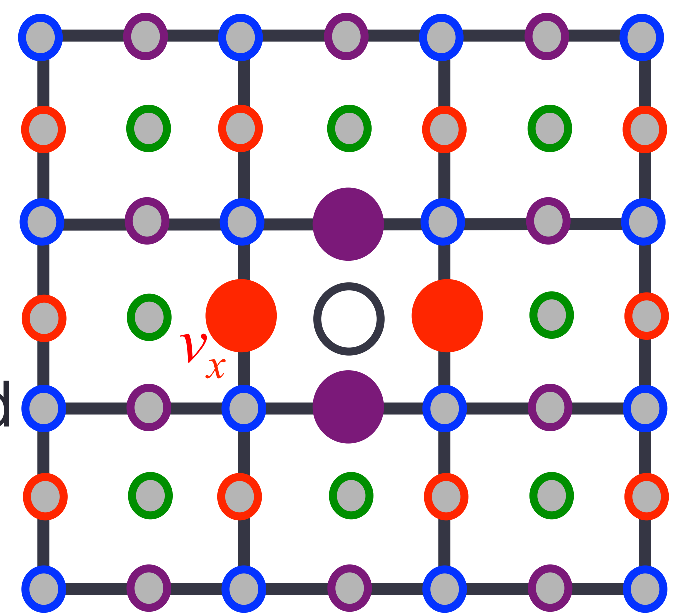
Pressure gradient discretisation is centred around central v_z point.

Staggered grids: continuity equation

$$\frac{v_{x,right} - 2v_{x,central} + v_{x,left}}{\Delta x^2} + \frac{v_{x,top} - 2v_{x,central} + v_{x,bottom}}{\Delta z^2} - \frac{P_{right} - P_{left}}{\Delta x} = 0$$

$$\frac{v_{z,right} - 2v_{z,central} + v_{z,right}}{\Delta x^2} + \frac{v_{z,bottom} - 2v_{z,central} + v_{z,top}}{\Delta z^2} - \frac{P_{bottom} - P_{top}}{\Delta z} = Ra\frac{T_{right} + T_{left}}{2}$$

$$\frac{v_{x,right} - v_{x,left}}{\Delta x} + \frac{v_{z,bottom} - v_{z,top}}{\Delta z} = 0$$

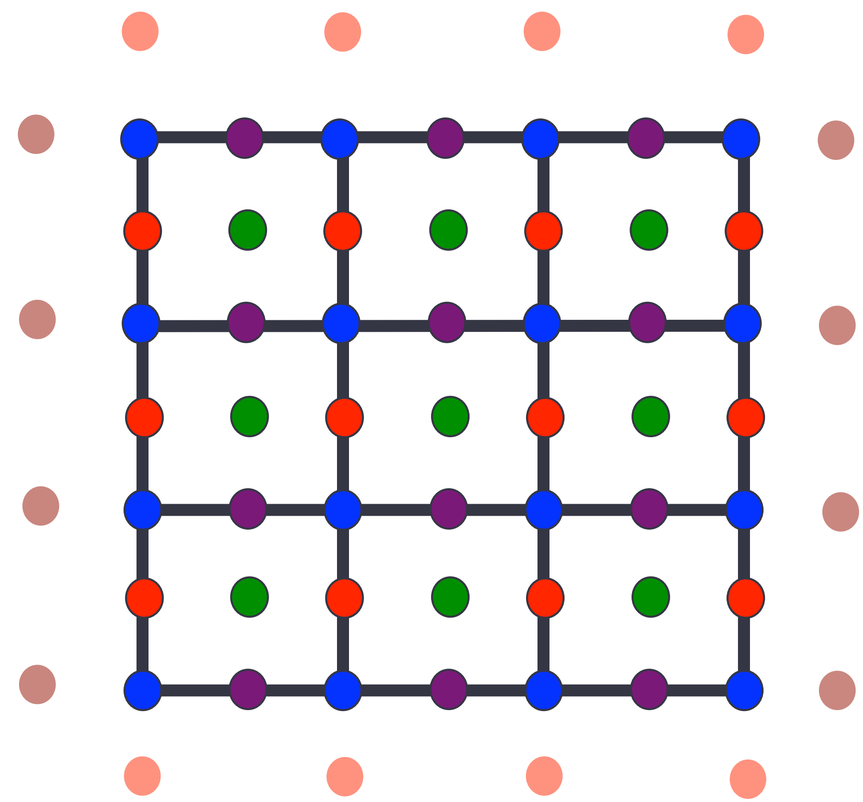


Both velocity gradients are centred around a common point

Staggered grids: using ghost points

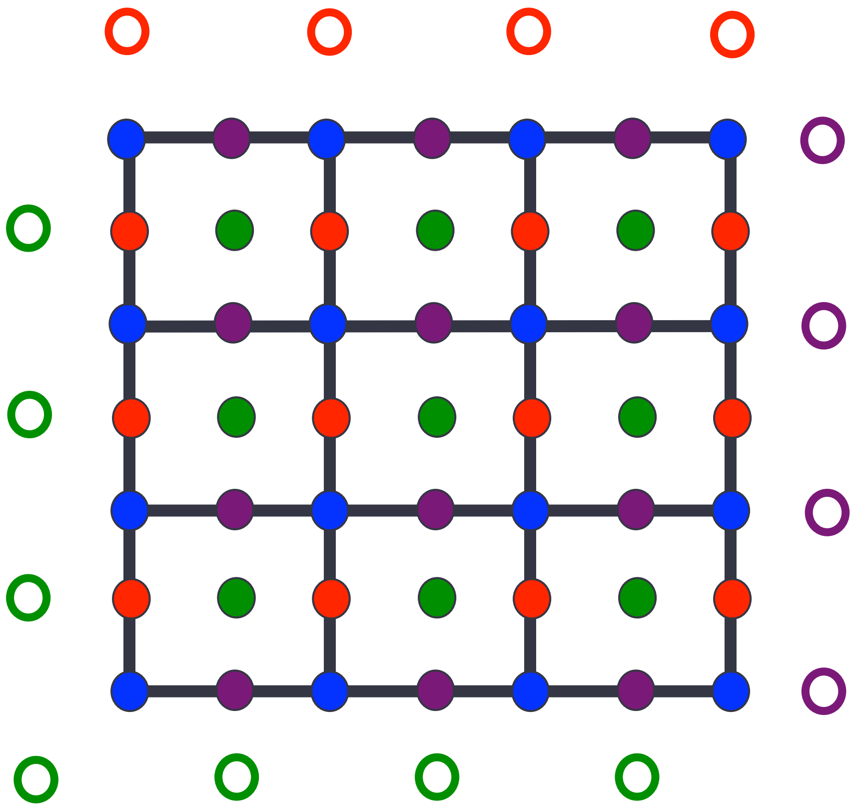
T v_z v_x P

Adding ghost points to allow for natural boundary conditions.
P, v_x and v_z.
But remember, these are no real points, and values are copied from internal points.



Staggered grids: adding unused points

T v_z v_x P



Adding unused points to make nr of grid points the same for P, v_x and v_z.
So nothing will be solved there, but these make the system nicely Structured.

Staggered grids: numbering grid points

T v_z v_x P

1. Continuity 1
2. Stokes-x 1
3. Stokes-z 1
4. Continuity 2
5. Stokes-x 2
6. Stokes-z 2
7. ….